THE SECRETARY PROBLEM SOLUTION DETAILS

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1. The Problem

You are in need of a secretary. You have time to interview N candidates, and you only are interested in selecting the best one, i^*. You are able to rank each candidate interviewing relative to the ones that you've seen by that point, and no candidate can have the same relative ranking (e.g. no ties for 2nd). You know that each candidate is completely independent and whose ranking has no bearing on the quality of those that will follow her. At each interview, i, you have the opportunity to accept the candidate, in which case you may not continue viewing the rest of the $N - i$ candidates, or you may reject him and move on to $i + 1$, but you will never have the option of returning to i; once a candidate is rejected she is rejected for life. **What is the selection strategy that will *maximize* the probability of selecting the candidate that is, in fact, best out of all potential N?**

2. How to Think

The first step in solving the problem is making the realization that the optimal strategy must occur as a type of Stopping Time rule.

> A Stopping Time with respect to a sequence of random variables, in this case $i_1, i_2, i_3, ...,$ is a random variable, τ, with the property that for each t, the occurrence or non-occurrence of the event $\tau = t$ only depends on the values of $i_1, i_2, i_3, ..., i_t$, and furthermore $P(\tau < \infty) = 1$; or in other words, τ is almost surely finite.

Stopping Times are used in decision theory to decide when a process should be stopped or continued based upon the data observed thus far.

One can see why the solution to our problem must be a non-trivial Stopping Time rule by considering the alternative strategy. The other type of strategy that can exist that is technically a Stopping Time rule as well, but is a trivial case is:

> Always pick the i^{th} candidate for some predetermined $i \in [1, N]$. $P(Success) = 1/N$

We can do much better than $1/N$ by applying the following rule, which yields the optimal solution:

Date: May 3, 2010.

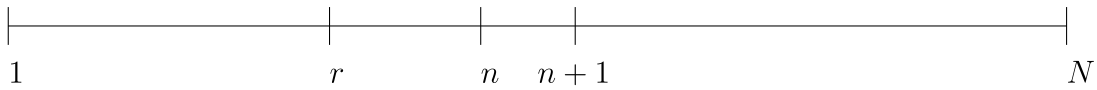
> Interview and reject the first r applicants, for $r < N$. Accept the very next applicant that is better than all the first r you interviewed.
> $P(Success) = P(r)$

We will now show that the optimal solution is found by optimizing $P(r)$ by the standard route of solving:

$$P'(r) = 0$$

3. Solution

The following diagram will be helpful to visualize the problem:



Let r be the last applicant you will see before you actually start considering hiring anyone (the last one you're going to reject no matter what). Let the best applicant of all N, i^* occur arbitrarily at $n+1$, and N is the total number of applicants you have the potential to interview. We then can say that i^* will not be chosen unless both of the following conditions are met:

(1) $n \geq r$
(2) The highest applicant in $[1, n]$ is the same highest applicant in $[1, r]$

The probability of this happening for some given n is

$$\frac{r}{n}\frac{1}{N}$$

This basically stems from the fact that the probability of i^* occuring at $n+1$ is $1/N$ and the probability of condition (2) is r/n. We can obtain $P(r)$ by summing over all possible $n \geq r$:

$$P(r) = \frac{1}{N}\left[\frac{r}{r} + \frac{r}{r+1} + \frac{r}{r+2} + \ldots + \frac{r}{N-1}\right] = \frac{r}{N}\sum_{n=r}^{N-1}\frac{1}{n}$$

This is great, and now we're ready to do some real magic. With a clever rearrangement, we notice that $P(r)$ is in fact a Riemann approximation to an intergral. By inspecting the expression in the limit as $N \to \infty$, letting $\lim_{N\to\infty}\frac{r}{N} = x$, and $\lim_{N\to\infty}\frac{n}{N} = t$ we find the following:

$$P(r) = \lim_{N\to\infty}\frac{r}{N}\sum_{n=r}^{N-1}\frac{N}{n}\frac{1}{N} = x\int_x^1\frac{1}{t}dt = -x\ln x$$

So in the limit as N grows infinitely large, we find that the ratio of applicants reviewed and rejected to the number of total applicants approaches x. We see then that solving $P'(r) = 0$ for r gives us the optimal ratio and the probability of success $P(r_{optimal})$.

$$P'(r) = -\ln x - 1 = 0 \implies x = \frac{1}{e}$$
$$P\left(\frac{1}{e}\right) = \frac{1}{e}$$
$$\frac{1}{e} \approx .37$$

The ratio of r to N is optimal at $\frac{1}{e}$ yielding a probability of success of, coincidentally, $\frac{1}{e}$ as well. So for $N >> 1$ the $r_{optimal}$ is nearly $\frac{N}{e}$, otherwise it can be found by computing $P(r)$ directly.

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